An Even More Straightforward Proof of Descartes's Circle Theorem

Alden Bradford

Descartes's circle theorem states that the radii of four mutually tangent circles r_1, r_2, r_3, r_4 satisfy

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right).$$

The radii are defined as negative if the corresponding circle encloses the others. In this way, we preserve the relation $d_{ij}^2 = (r_i + r_j)^2$, where d_{ij} is the distance between the centers of circles r_i and r_j.

An article [7] published in this journal in 2019 gives a short history of the theorem and provides an original and straightforward proof based on Heron's formula. Here, we provide an even more straightforward proof based on a generalization of Heron's formula, the Cayley–Menger determinant.

Cayley–Menger Determinants

The Cayley–Menger determinant was first introduced by Arthur Cayley in 1841 [2, 8]. It gives a formula for the volume of an n-simplex in terms of the pairwise distances between the vertices. In the case of a triangle with side lengths a, b, c and area A, the formula is equivalent to Heron's formula,

$$-16A^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

To prove Descartes's theorem, we will use the Cayley–Menger determinant for the tetrahedron of volume v,

$$288v^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{42}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

This formula is quite simple to prove. We present the tetrahedral case here, adapted from the more general proof given in [1] for an n-simplex. Let x_j be the jth vertex of the tetrahedron, and x_{ij} its ith component. Write

$$U = \begin{bmatrix} 1 & |x_1|^2 & |x_2|^2 & |x_3|^2 & |x_4|^2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{21} & x_{22} & x_{23} & x_{24} \\ 0 & x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

and

$$W = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Because $d_{ij}^2 = |x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2x_i \cdot x_j$, we have that $U^T W U = D$, the matrix of the Cayley–Menger determinant. On the other hand, $|W| = 8$, and we can expand along the first column of U to reach the standard cross-product-style formula for the volume of a tetrahedron,

$$|U| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{vmatrix} = 6v.$$

Putting these together yields $|D| = \left|U^T W U\right| = |U|^2|W| = 288v^2$.

Proof of the Circle Theorem

The strategy for our proof of Descartes's circle theorem is to consider the tetrahedron whose vertices are the centers of the given circles. We apply the Cayley–Menger determinant formula, replacing each d_{ij} with $r_i + r_j$. After simplifying, we have the formula

$$v^2 = \left(\frac{r_1 r_2 r_3 r_4}{3}\right)^2 \left[\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 - 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)\right].$$

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This equation gives us exactly what we need. If the four circle centers lie in a plane, then their tetrahedron will have zero volume. Since none of the radii are zero, the term $r_1 r_2 r_3 r_4$ must be nonzero, and hence

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2-2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)=0.$$

That is the whole of the proof. All that remains is to justify the volume formula given above.

Let D be the matrix in the Cayley–Menger determinant formula. When we expand the terms $d_{ij}^2=r_i^2+2r_ir_j+r_j^2$, we are left with the term r_i^2 repeated along each row. We can eliminate it using the matrix

$$P=\begin{bmatrix}1 & -r_1^2 & -r_2^2 & -r_3^2 & -r_4^2\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

giving us

$$P^TDP=\begin{bmatrix}0 & 1 & 1 & 1 & 1\\ 1 & -2r_1^2 & 2r_1r_2 & 2r_1r_3 & 2r_1r_4\\ 1 & 2r_2r_1 & -2r_2^2 & 2r_2r_3 & 2r_2r_4\\ 1 & 2r_3r_1 & 2r_3r_2 & -2r_3^2 & 2r_3r_4\\ 1 & 2r_4r_1 & 2r_4r_2 & 2r_4r_3 & -2r_4^2\end{bmatrix}.$$

Each column and row has a common factor of r_i, so we can pull it out using the matrix $Q=\operatorname{diag}(1,1/r_1,1/r_2,1/r_3,1/r_4)$. Then

$$Q^TP^TDPQ=\begin{bmatrix}0 & \frac{1}{r_1} & \frac{1}{r_2} & \frac{1}{r_3} & \frac{1}{r_4}\\ \frac{1}{r_1} & -2 & 2 & 2 & 2\\ \frac{1}{r_2} & 2 & -2 & 2 & 2\\ \frac{1}{r_3} & 2 & 2 & -2 & 2\\ \frac{1}{r_4} & 2 & 2 & 2 & -2\end{bmatrix}.$$

Write $R=\begin{bmatrix}1/r_1 & 1/r_2 & 1/r_3 & 1/r_4\end{bmatrix}^T$ and $S=2\mathbb{1}\mathbb{1}^T-4I$, where $\mathbb{1}=\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$, which allows us to put this in the block form

$$Q^TP^TDPQ=\begin{bmatrix}0 & R^T\\ R & S\end{bmatrix}.$$

There are a couple of things to note about S. First, observe that $S^2=16I$, and so $S^{-1}=\frac{1}{16}S$. Second, we can readily compute $|S|=-256$. We take advantage of both of these when applying a common rule for the determinants of block matrices. Recall that if A_{22} is invertible, then

$$\begin{vmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}=|A_{22}|\left|A_{11}-A_{12}A_{22}^{-1}A_{21}\right|.$$

We apply this here to obtain

$$\begin{aligned}|Q^TP^TDPQ| &= -|S|R^TS^{-1}R\\ &= 16R^TSR\\ &= 32R^T[\mathbb{1}\mathbb{1}^T-2I]R\\ &= 32\left[(R^T\mathbb{1})(\mathbb{1}^TR)-2R^TR\right]\\ &= 32\left[\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2\right.\\ &\quad\left.-2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)\right].\end{aligned}$$

On the other hand,

$$|Q^TP^TDPQ|=|P|^2|Q|^2|D|=(1)^2\left(\frac{1}{r_1r_2r_3r_4}\right)^2\left(288v^2\right).$$

This completes the proof.

Generalizing to Higher Dimensions

The Soddy–Gosset theorem generalizes the Descartes circle theorem to configurations of $n+2$ mutually tangent spheres in n dimensions. With $r_1, r_2, \ldots, r_{n+2}$ denoting the signed radii of the n-dimensional spheres, the theorem states that

$$\left(\frac{1}{r_1}+\cdots+\frac{1}{r_{n+2}}\right)^2=n\left(\frac{1}{r_1^2}+\cdots+\frac{1}{r_{n+2}^2}\right).$$

The above proof generalizes perfectly well to higher dimensions, giving exactly the Soddy–Gosset theorem. The only changes are the sizes of the matrices. In general, the Cayley–Menger determinant for $n+2$ points evaluates to

$$(-1)^n2^{n+1}\left((n+1)!v_{n+1}\right)^2,$$

where v_{n+1} is written to emphasize that the formula gives an $(n+1)$-dimensional volume. The general matrix S has determinant and inverse

$$\begin{aligned}|S| &= (-1)^{n+1}2^{2n+3}n,\\ S^{-1} &= \frac{1}{4n}\mathbb{1}\mathbb{1}^T-\frac{1}{4}I.\end{aligned}$$

Carrying these changes through the computation gives

$$\begin{aligned}v_{n+1}^2 =& 2^n\left(\frac{r_1\cdots r_{n+2}}{(n+1)!}\right)^2\left[\left(\frac{1}{r_1}+\cdots+\frac{1}{r_{n+2}}\right)^2\right.\\ &\left.-n\left(\frac{1}{r_1^2}+\cdots+\frac{1}{r_{n+2}^2}\right)\right].\end{aligned}$$

Analysis

The proof above has an intuitive meaning, since it connects the Descartes formula to the volume of a simplex. It generalizes nicely to higher dimensions as well. Since Descartes's theorem deals only with the distances between points, it seems natural to approach this problem from the perspective of distance geometry. Cayley–Menger determinants are a foundational tool in distance geometry. In Cayley's original paper he used the argument that the determinant should be zero when all the points lie in a lower-dimensional subspace, which is the same way we use it here. The simplex that joins the centers in a sphere packing has been used before to prove facts about sphere packings [3, 4, 9]. Certainly, many brilliant minds have sat down to prove Descartes's theorem, bringing all manner of advanced techniques to bear.

Given all these hints littered throughout history, why wasn't this proof noticed 181 years ago when Cayley first introduced his determinant? We can only speculate, of course, but I would suggest the reason is that there was less interest in Descartes's circle theorem back then. We are in the middle of a renaissance of interest in the theorem, due both to the complex-valued generalization [6] and interest in the number-theoretic properties of Apollonian circle packings [10]. The proof given here does not address the complex-valued generalization, which is the main limitation of this proof. There is a concise and elegant proof of that due to Jerzy Kocik [5].

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Alden Bradford, Department of Mathematics, Purdue University, 150 N. University St., West Lafayette, IN 47907, USA. E-mail: bradfoa@purdue.edu

References

[1] Marcel Berger. *Geometry*. Springer, 1987.

[2] Arthur Cayley. On a theorem in the geometry of position. *Cambridge Mathematical Journal* 2 (1841), 267–271.

[3] Hao Chen. Apollonian ball packings and stacked polytopes. *Discrete & Computational Geometry* 55:4 (2016), 801–826.

[4] Daryl Cooper and Igor Rivin. Combinatorial scalar curvature and rigidity of ball packings. *Mathematical Research Letters* 3:1 (1996), 51–60.

[5] Jerzy Kocik. Proof of Descartes circle formula and its generalization clarified. Available at https://doi.org/10.48550/arXiv.1910.09174, 2019.

[6] Jeffrey C. Lagarias, Colin L. Mallows, and Allan R. Wilks. Beyond the Descartes circle theorem. *American Mathematical Monthly* 109:4 (2002), 338–361.

[7] Paul Levrie. A straightforward proof of Descartes's circle theorem. *Mathematical intelligencer* 41:3 (2019) 24–27.

[8] Leo Liberti and Carlile Lavor. Six mathematical gems from the history of distance geometry. *International Transactions in Operational Research* 23:5 (2016), 897–920.

[9] Igor Rivin. An extended correction to "combinatorial scalar curvature and rigidity of ball packings" (by D. Cooper and I. Rivin). Available at https://doi.org/10.48550/arXiv.math/0302069, 2003.

[10] Peter Sarnak. Integral Apollonian packings. *American Mathematical Monthly* 118:4 (2011), 291–306.